Introduction to Pure Mathematics*

CDT Lecture Notes

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Last Updated: October 7, 2023

1 Set Theory

Definition 1. *A* ***set*** *is a collection of objects.*

Theorem 1. *The following set is well-defined*

$$S = \{x, 42, 1.1234, \square, \otimes, \textit{blue}\}. \tag{1}$$

Proof. Follows directly from the definition $\square$

This example was chosen to illustrate the fact that sets can really contain anything. Some of the most important examples of sets you will encounter are the different kinds of **numbers**

$$\mathbb{N} = \{0, 1, 2, 3, 4, \cdots\} \quad \text{(Natural Numbers)} \tag{2}$$

$$\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\} \quad \text{(Integers)} \tag{3}$$

but again, our sets can contain anything, not just numbers. Note that in this course we will not count 0 as a natural number, $0 \in \mathbb{Z}$ however. If I want to say that a particular object x is a member of a set S, I can say $x \in S$. Alternatively, if I want to say x is not a member of S then I can write $x \notin S$. Let us define a set to play with.

Definition 2. *Let $\mathcal{P}$ be the set of regular polygons, i.e.*

$$\mathcal{P} = \{\triangle, \square, \cdots\} \tag{4}$$

We can then make statements like $\triangle \in \mathcal{P}$ and $\bigcirc \notin \mathcal{P}$. Another important thing we can do with sets is look at *subsets*, some collection of objects from a set which we treat as a set itself. If I want to say a set X is a subset of Y then I can write $X \subset Y$, or $X \subseteq Y$ if I want to allow X and Y to also be equal. Some good examples of subsets are the even and odd numbers

$$\mathbb{N}_{even} = \{2, 4, \cdots\} \subset \mathbb{N} \tag{5}$$

$$\mathbb{N}_{odd} = \{1, 3, 5, \cdots\} \subset \mathbb{N} \tag{6}$$

but I can equally well talk about the subset of $\mathcal{P}$ with an even number of edges: $\{\square, \cdots\} \subset \mathcal{P}$.

Definition 3. *Let X and Y be sets. We say X and Y are equivalent, i.e. the same set, if and only iff they are subsets of each other. That is*

$$X \subseteq Y \text{ and } Y \subseteq X \iff X = Y \tag{7}$$

*Based on "*Introduction to Topology*" by Mark R. Dennis

1.1 Operations on Sets

Having sets is all well and good, but what can we do with them. Two useful and very natural things we can do with sets are glue them together and look at how they overlap, we call these the *union* and *intersection* of the sets.

Definition 4. *Let X and Y be sets. The* ***union*** *of X and Y, written $X \cup Y$ is the set of all the elements of X and Y. Or, in set-builder notation*

$$X \cup Y := \{z \mid z \in X \text{ or } z \in Y\}. \tag{8}$$

If you have not come across set builder notation before then you can think of it as a way of defining sets from other sets using a rule instead of specifying the elements explicitly (which can get a bit exhausting if your set has infinitely many members). You should read (8) as

$X \cup Y$ is the set of all elements z such that z is a member of X **or** z is a member of Y.

So now we know how to join two sets using a union we will define the next natural operation

Definition 5. *Let X and Y be sets. The* ***intersection*** *of X and Y, written $X \cap Y$ is the set of all the elements of X and Y. Or, in set-builder notation*

$$X \cup Y = \{z \mid z \in X \text{ and } z \in Y\}. \tag{9}$$

Again, translating the set-builder notation into English would read...

$X \cap Y$ is the set of all elements z such that z is a member of X **and** z is a member of Y.

Using these ideas we can make basic statements like $\{1,2\} \cup \{3,4\} = \{1,2,3,4\}$ and $\{1,2,3,4\} \cap \{3,4,5,6\} = \{3,4\}$. Or maybe some more interesting statements like $\mathbb{N}_{odd} \cup \mathbb{N}_{even} = \mathbb{N}$ and $\mathbb{N}_{odd} \cup \mathbb{N}_{even} = \emptyset$. In this last one, we know that all the natural numbers are odd or even, but none are both, so their intersection is empty. We use the special symbol $\emptyset = \{\ \}$ to denote the set with no elements. Note that for any set A, we always have that $\emptyset \subseteq A$. Using what we have learned we can also define another important set of numbers the *rationals*

Definition 6. *The* ***rational numbers****, denoted $\mathbb{Q}$ are the numbers which can be written as the ratio of whole numbers. In set builder notation this is*

$$\mathbb{Q} := \left\{ \frac{A}{B} \ \forall\, A \in \mathbb{Z} \text{ and } B \in \mathbb{N} \right\} \tag{10}$$

To help you think about these three different types of number we have defined, you could try proving the following

Theorem 2. *The following is a true statement*

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \tag{11}$$

Exercise 1. *Prove theorem 2*

The last operation we will define in this section will be in idea of *subtraction.*

Definition 7. *Let X and Y be sets. The* ***subtraction*** *$X - Y$, read as "X minus Y", is given by*

$$X - Y := \{x \in X \mid x \notin Y\}, \tag{12}$$

which can be read as the set of elements of X which are ***not*** *members of Y.*

One of the first things you will do in the exercise sheets is investigate for yourself what properties these operations have and how they interact with one another, e.g. can you mix them about like you can regular multiplication.

1.2 Abstraction of Membership

The previous section has described what set theory is using definitions, but what is a set *really*? In my opinion a set is an *abstraction* of the concept of *membership*. Given a set S and some object x remember that the only information I can possibly obtain about these two objects is whether or not $x \in S$ or $x \notin S$. Consider again the following set

$$S = \{x, 42, 1.1234, \square, \otimes, \textit{blue}\}. \tag{13}$$

Despite us having this idea that 42 is a number or *blue* is a color, to S these objects are it's members and the only property that 42 and *blue* have are that they are members of S, nothing more. We will often label elements of sets with symbols like numbers because they refer to some other structure above and beyond that of the concept of membership. For example, consider the set

$$S = \{5, 6, 7, 8\}. \tag{14}$$

In strict set-theoretic terms there is no sense here in which $7 > 5$ or $6 < 8$, what we have is a set containing four elements which we have chosen to depict using some symbols we call numbers. In a similar vein the statement $5 + 6$ has no meaning in S, addition is an operation that we do on numbers and not set elements.

But we should not forget that the idea of set theory is to abstract away the internals of some messy object and treat it by whether or not is it is a member of a particular set (for example, the set of numbers with a particular property). In the language of set theory questions about membership are computed by taking unions $\cup$ and intersections $\cap$ of different sets. This is one reason people find it helpful to think of sets and their manipulations in terms of Venn diagrams.

This discussion is actually very general and encapsulates much of the process of *doing* mathematics. We construct some structure which carefully manifests some feature in an abstract manner (membership in the case of set theory) and allows you to manipulate this feature using some operations. One of the first steps in tackling a mathematical problem is deciding on what the *correct abstraction* is to use to describe it; does it throw away too much information or too little? Does the calculation you want to do appear contrived and forced or flow naturally? If you have chosen your abstraction well then the operations will appear very *natural*, as unions and intersections do in set theory. In terms of this course, the abstraction we are studying, that of *topology* is that of a sense of *shape*. There are other abstractions which capture the idea of *shape* in different ways (homotopy, homology, manifolds and moduli spaces to name a few), but the questions that are of interest to us are most naturally answered with topology. Below are some examples of some sets with some nice extra structure.

1.3 The Continuum and Intervals

When it comes to *topology*, there is one last set of numbers that is of the upmost importance, the *real numbers*, which we give the symbol $\mathbb{R}$. Whenever you have worked with measuring lengths, the real numbers were the numbers you were using, even if you didn't know it, quite often we will refer to $\mathbb{R}$ as the *real line*. They include the rational numbers as well as numbers like π and $\sqrt{2}$. They are the numbers that you have been using your whole life. When it comes to topology the concept of $\mathbb{R}$ and the ways that we can "continuously" transform them are of central importance. Unfortunately a precise definition of the real numbers is quite technical and will be left to later in the course. As an abstraction $\mathbb{R}$ captures the notion of a continuum.

Quite often we will want to work with specific subsets of the real line called intervals. We use the notation $(1, 5)$ to denote the subset of $\mathbb{R}$ containing all numbers between 1 and 5 (exclusively) and $[1, 5]$ if we want to talk inclusively. Concretely

Definition 8. *Let a and b be real real numbers. We call*

$$(a, b) := \{r \in \mathbb{R} \mid a < r < b\} \tag{15}$$

the ***open interval*** *from a to b*

Definition 9. *Let a and b be real real numbers. We call*

$$[a, b] := \{r \in \mathbb{R} \mid a \leq r \leq b\} \tag{16}$$

the ***closed interval*** *from a to b*

Alternatively you can think of $[a, b]$ as a line segment we have extracted in $\mathbb{R}$ which includes its endpoints and (a, b) as a line segment in $\mathbb{R}$ which does not include its endpoints. We can convert between these by adding and removing the appropriate points, i.e. $(a, b) = [a, b] \setminus \{a, b\}$ and $[a, b] = (a, b) \cup \{a, b\}$. This basic idea of a object being *open* or *closed* will be very important when we come to study topology. You will get practice manipulating intervals and determining if they are *open* or *closed* in the exercise sheets. The other form of interval you will encounter is one which includes only one of its endpoints and excludes the other.

Definition 10. *Let $a, b \in \mathbb{R}$. A* ***half-open*** *or* ***half-closed*** *(depending on how optimistic you are feeling) is an interval of the form $(a, b]$ or $[a, b)$*

A note on convention. Quite often we will want to think about intervals that include the rest of $\mathbb{R}$ after a certain number, for example the set of all strictly positive real numbers. As a set we call this $\mathbb{R}_+$ and can write it as the interval

$$\mathbb{R}_+ = \{r \in \mathbb{R} \mid r > 0\} = (0, \infty). \tag{17}$$

In fact it can often be useful to think of $\mathbb{R}$ itself as the interval $\mathbb{R} = (-\infty, \infty)$. By convention any interval which goes to infinity is written as an *open interval*.

1.4 The $\mathcal{U}$niverse

To discuss more sophiticated set-theoretic topics we need to introduce the concept of the *universal set*, denoted $\mathcal{U}$. Formally, $\mathcal{U}$ is defined as the set of all elements, including itself. As Russel showed with his famous paradox, $\mathcal{U}$ is not an object which technically exists, but for our purposes (so-called Naive set theory), it is useful to think of $\mathcal{U}$ as some universe in which all possible elements of a set live (remember that elements of sets can be sets themselves). For our purposes we could define the behaviour of $\mathbb{U}$ operationally as

Definition 11. *Let A be any set, the* ***universal set,*** *$\mathcal{U}$ is the set such that*

$$A \subset \mathcal{U} \tag{18}$$

is always true.

In practice we will usually take $\mathcal{U}$ to be a space like $\mathbb{R}^n$. One might rightfully ask why such an esoteric and logically unsound object has been conjured up, here is your answer

Definition 12. *Let X be a set, we call X^c the* ***complement*** *of X, its elements are given by*

$$X^c = \{x \in \mathcal{U} \mid x \notin X\}. \tag{19}$$

It is useful to think of the complement as the *inverting* or *turning-inside-out* of a set. For example, let us take $\mathcal{U} = \mathbb{R}$ (or equivalently we shall say we are taking *complements over* $\mathbb{R}$ and compute the complements of some intervals.

$$\mathbb{R}^c = \emptyset \tag{20}$$
$$\emptyset^c = \mathbb{R} \tag{21}$$
$$(a, b)^c = (-\infty, a] \cup [b, \infty) \tag{22}$$
$$\{3\}^c = (-\infty, 3) \cup (3, \infty) \tag{23}$$

Don't worry if you don't immediately see why these are true, you will practice taking complements and developing an intuition for all the set-theoretic operations while doing the exercises.

2 Functions

Having sets and being able to manipulate them is all well and good, but what we want to do next is build relationships between different sets. We do this using *functions*. Functions are objects that we construct that we use to relate elements of two different sets. Let X and Y be sets. We say f is a *function* from X to Y, which we write $f : X \to Y$, and it assigns to every element $x \in X$ a unique member of y, typically written $f(x) \in Y$. We can draw a little diagram to represent this.

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \cup\!\!| & & \cup\!\!| \\ x & \longmapsto & f(x) \end{array}$$

We need two pieces of information to define a function; the sets it moves between (the top arrow) and the rule for transforming the elements (the bottom arrow). Where the function starts, X we call the **domain**, where is maps to, Y, the **codomain**. Quite often if X and Y are sets of numbers we can write the function rule as an equation, i.e. $f(x) = x + 1$. For a non-equation example, we could introduce the edge-counting function $Edge : \mathcal{P} \to \mathbb{N}$, where the rule is $Edge(p) = n$, n being the number of edges of p, giving results like $Edge(\square) = 4$. To put all these parts into a picture

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\quad Edge \quad} & \mathbb{N} \\ \cup\!\!| & & \cup\!\!| \\ p & \longmapsto & Edge(p) \end{array}$$

While the idea of a function as some sort of black box which performs an operation on whatever number you feed into it should be familiar, I want to take a more general and pictorial approach. Quite often people will use words like *mapping* or *transformation* as synonyms for *function* and it is this more geometric flavour I want to focus on. First though, we need one final definition.

Definition 13. *Let X and Y be sets and $f : X \to Y$ be a function. The **image** of f, denoted $f(X)$ is the set*

$$f(X) = \{f(x) \ \forall \ x \in X\} \tag{24}$$

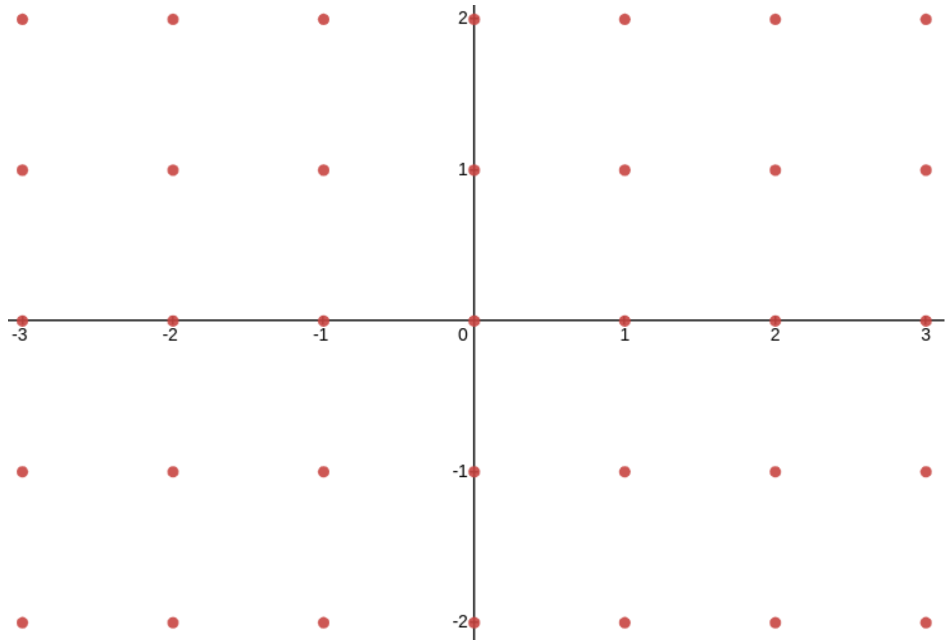


Figure 1: $\mathbb{Z}^2$ highlighted as a subset of $\mathbb{R}^2$

2.1 Curves As Functions

To be able to talk about geometries it is useful to work in 2-dimensional space (how many interesting shapes can you draw in 1 dimension). To build the *piece of paper* upon which we are to affect our artistry we need a new set-theoretic operation on sets.

Definition 14. *Let A and B be sets. The **Cartesian product** of A and B , written $A \times B$ is the set*

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in b\} \tag{25}$$

where (a, b) denotes an not an open interval but an ordered pair (for the moment you can think of this as a special kind of set where the ordering matters, i.e. $(a, b) \neq (b, a)$).

Following this definition we construct $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, which we will refer to as *the plane*. Elements of this set are an ordered pair of real numbers, a *point on the plane* which has x and y coordinates. We can form other Cartesian products, for example, Figure 1 shows how $\mathbb{Z}^2$ looks as a subset of $\mathbb{R}^2$, forming a lattice pattern in the plane. There are other subsets of $\mathbb{R}^2$ that you have encountered before

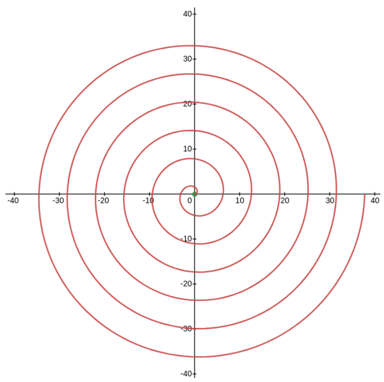
Definition 15. *Let $r \in \mathbb{R}_+$. The **circle**, denoted S^1, is the subset of $\mathbb{R}^2$ given by*

$$S^1 = \{(a, b) \in \mathbb{R}^2 \mid a^2 + b^2 = r^2\}. \tag{26}$$

Now that we have the canvas, we can paint a picture. The idea I want to get across here is how one can formalise the idea of *drawing a shape/line* in $\mathbb{R}^2$ as a function.

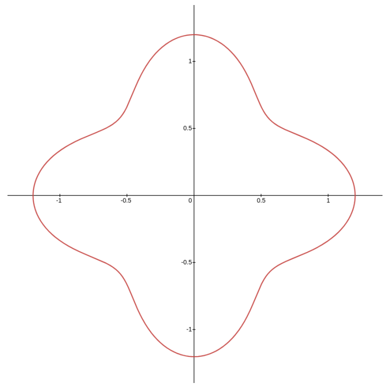
Definition 16. *Let $t \in \mathbb{R}$ and*

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\gamma} & \mathbb{R}^2 \\ \cup & & \cup \\ t & \longmapsto & \gamma(t) \end{array}$$



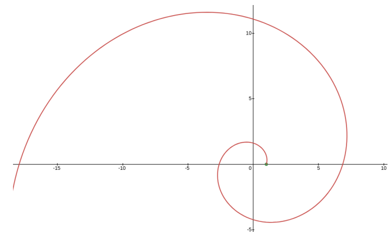
(a) The Archimedean spiral[a], given by the image of $\gamma(t) = (t\cos(t), t\sin(t))$

[a] https://www.desmos.com/calculator/tyip0te0nq



(b) An oscillating circle[a], given by the image of $\gamma(t) = \left(1 + \frac{1}{5}\cos(4t)\right)(\cos(t), \sin(t))$

[a] https://www.desmos.com/calculator/1hf4vh4agj



(c) The golden spiral[a], given by the image of $\gamma(t) = \varphi^{\frac{2t}{\pi}}(\cos(t), \sin(t))$

[a] https://www.desmos.com/calculator/3ajs88qv1g

Figure 2: Three variations on the circle

be a continuous (for now your intuition suffices but we will go on to define properly what continuity means soon) function. We call γ a ***curve*** *in $\mathbb{R}^2$.*

Now what does gamma do? It takes some number t and gives us a point in the plane. We can then imagine feeding successive values of t into γ and watching the path that is traced out in $\mathbb{R}^2$. Need to define what am image is further up... For a concrete example...

Definition 17. *Let $r \in \mathbb{R}$ and $t \in [0, 2\pi] \subset \mathbb{R}$. The* ***circle****, denoted S^1, is the image of the curve*

$$\gamma(t) = (r\cos(t), r\sin(t)) \tag{27}$$

You might find it useful to play around with different ways of drawing the circle[1]. You can make a lot of nice curves with interesting properties by playing around with some some of the functions you already know, Figure 2 gives an example of some of the spiral curves that you can make by making small changes to our circle curve

2.2 Anatomy of a function

When we have a function between some sets, there are some special names that are given to particular set related to the function

Definition 18. *Let X and Y be sets and $f : X \to Y$ be a function. We then identify the following special sets:*

- *The set X is called the* ***domain*** *of f*
- *The set Y is called the* ***co-domain*** *of f*
- *The image of X under f, $f(X)$, is called the* ***range*** *of f*

There is a subtlety here that the range and codomain of a function are not always the same set. For example, the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$ has domain and codomain $\mathbb{R}$, but the range of f is only the *positive reals* $\mathbb{R}_{\geq 0} = [0, \infty)$

[1] https://www.desmos.com/calculator/zduh1psizf

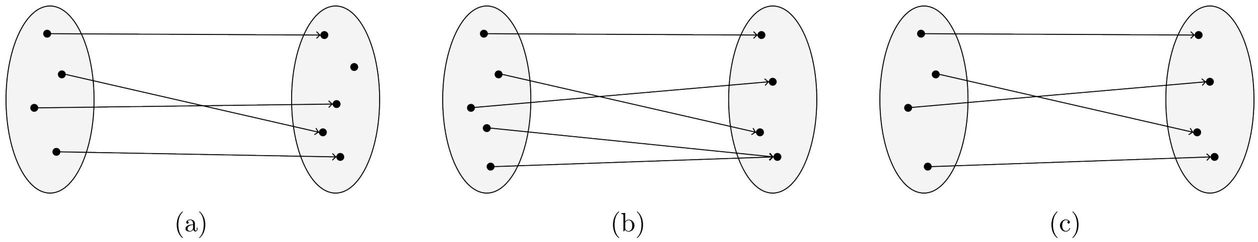


Figure 3: Examples of a function mapping between two sets in an (a) injective, (b) surjective and (c) bijective manner.

2.3 The Pre-Image

The function *Edge* as defined above is particularly nice because it has an *inverse*. Usually for a function f we write its inverse as f^{-1}. In this case the behaviour of $Edge^{-1}$ is fairly easy to see

$$Edge^{-1}(\triangle) = \{3\}\ , \quad Edge^{-1}(\square) = \{4\}\ , \ \cdots \tag{28}$$

For reasons which will become clear, the preimage is a set, $Edge^{-1}(\triangle)$ is the set containing 3. If the preimage is particularly well-behaved then we can construct an ...

Definition 19 (Inverse Functions)**.** *Let X and Y be sets and $f : X \to Y$ be a function. If we have a function $g : Y \to X$ which satisfies*

$$g(f(x)) = x \quad \forall x \in X \tag{29}$$

*then we call g the **inverse** of f and write $g = f^{-1}$*

There are two special types of functions which have particular properties

Definition 20 (Injectivity)**.** *A function is **injective** if the preimage of each point in the range is a singleton*

Definition 21 (Surjectivity)**.** *A function is **surjective** if the preimage of each point in the codomain is nonempty.*

Definition 22 (Bijectivity)**.** *A function which is both injective and surjective is called **bijective**.*

These definitions are useful because of the following fact, if $f : X \to Y$ is a bijective function, then there must exist a function f^{-1} which is the inverse of f, that is, a function is invertable if and only if it is bijective. Once you understand what sur/bi/in-jectivity means, you will find that there are lots of different ways of stating these properties, the definitions I have given are my preferred ones, you are welcome to find and use your own[2]. Figure 3 shows who you can think of these poperties in terms of simple set mappings.

2.4 Functions of multiple variables

It is possible to recast the operations of familiar everyday arithmetic as functions of two variables. To convince you of this, here are some quick-fire functional definitions of arithmetic.

Definition 23. *Let $x, y \in \mathbb{R}$. We call **addition**, denoted by $+$, the function*

$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R} : (x, y) \mapsto +(x, y) = x + y. \tag{30}$$

Definition 24. *Let $x, y \in \mathbb{R}$. We call **multiplication**, denoted by $\cdot$, the function*

$$\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R} : (x, y) \mapsto \cdot(x, y) = x \cdot y. \tag{31}$$

[2] A good place to start looking is `n.wikipedia.org/wiki/Bijection`

3 A little Abstract Algebra

Now that we have discussed some of the formalities of mathematics we can start a little mathematical adventure of our own, by building and investigating a new structure.

3.1 Properties of binary functions

Now that we have seen some examples of functions with two arguments, what we will now call a *binary operation*, we can abstract this idea and try and study binary operations in general.

Definition 25. *Let S be a set. We call a function*

$$\circ : S \times S \to S : (x, y) \mapsto x \circ y \tag{32}$$

*a **binary operation** on S. Alternatively, $\circ$ is a binary operation on S if it is closed, that is,* $\text{range}(\circ) \subseteq S$.

Theorem 3. *The functions $+$ and $\cdot$ we defined earlier are examples of binary operations over $\mathbb{R}$*

There are a number of special properties that binary operations can have, the main ones we are interested in are defined below

Definition 26. *Let X be a set and $\circ : X \to X$ be a binary operation on X, we call $\circ$...*

- ***abelian** or **commutative** iff $x \circ y = y \circ x$ for all $x, y \in X$*
- ***associative** iff $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in X$*

Many of the operations you are familiar with, such as addition and multiplication over $\mathbb{R}$ are abelian and associative operations, have a think about other operations you know (on numbers or other mathematical objects you know from your discipline) and see if you can think of any examples of non-commutative or non-associative operations. Quite often in physics we end up working with sets that have a particularly nice binary operation which are called *groups*. If set theory is the natural abstraction of membership then the concept of a *group* is the natural incarnation of the concept of symmetry.

3.2 Mathematical Statements

Once you have read a bit of pure mathematics, you will notice that there is a similar style of presentation that many of them follow

- Define some new object
- Establish some basic facts about manipulations of the object
- Define some operations on the object, usually how one can make new version of objects from old ones
- Prove some nontrivial theorems about the objects

in the next section we are going to briefly discuss the idea of groups and symmetries following this pattern so you can see it in action (this is not necessarily the best way of writing mathematics, but the vast majority of reference books are written in this style). First though we are going to discuss each of these steps a little bit, as the motivation behind this workflow is what is most likely to separate the mathematics you are doing in this course from what you have done before.

If we are going to define some new mathematical object this usually involves the listing of a number of *axioms*, usually given in the set-theoretic language. This takes the form of a list of conditions, and we say that any set which satisfies those conditions is an example of that object. In this sense the axioms provide a starting point from which to construct the objects and will provide the basic logical framework that we will use to prove theorems about them. The axioms could be chose from scratch or we could take the axioms of a pre-existing object and loosen, tighten or otherwise modify them to create a different object. A famous example of a axiomatic system is Euclid's axiomatisation of planar geometry[3].

Next, basic facts about how the object works are then proved. You will see this in the exercise sheet when you prove various properties about how the set-theoretic operations we have looked at interact with each other. When we state some property of a mathematical object we call that a **proposition**. We then deploy the tools of creativity and logic to construct a **proof** of the statement, at which point the proposition is then referred to as a **theorem**. Important results are typically referred to as theorems, with smaller supporting technical arguments are referred to as **lemmas**.

Once the basic properties of the object have been worked out, the next thing to do is then start relating your new object to some other mathematical object (of course this becomes easier the more maths you know), for example if you created a new object by relaxing one of the axioms defining an existing object, you might expect that these new objects are manifestly examples of your new object.

There might be some facts about our object that we are convinced of but are not quite able to prove. If these problems are considered difficult and important enough then they are given the slightly grandiose title of *conjecture*. The Millenium Prize Problems are a famous set of conjectures with a $\$1,000,000$ prize for whoever can prove them.

Let us now see this process in action

3.3 Aspects of groups and symmetries

Definition 27. *Let G be a set and $\circ$ be a binary operation on G. We call $(G, \circ)$ a **group** if the following properties, known as the group axioms, are satsfied:*

- *(Identity) There exists a unique element $e \in G$, called the **identity**, such that $e \circ g = g \circ e = g$ for all $g \in G$*
- *(Inverse) For each $g \in G$ there exists a unique element $g^{-1} \in G$ called the **inverse** such that $g \circ g^{-1} = g^{-1} \circ g = e$.*
- *$\circ$ is **associative***

Note that some definitions of a group require that $\circ$ be closed, for us that is implied by our definition of a binary function.

As a first example, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ all form groups under addition. A notable property missing from the above is that we haven't required $\circ$ to be commutative. Can you think of an example of a non-commutative group?

Exercise 2. *Prove that the real numbers with 0 removed, $(\mathbb{R} - \{0\}, +)$, form a group under addition.*

[3] A beautiful copy of Euclid's book is available on the Internet Archive

Groups are simple enough objects that we don't need to immediately prove anything about their manipulations. When working with sets we saw that functions were what we used to map one set onto another, groups too are related by functions but we have to be careful that the function respects the group structure

Definition 28. *Let $(X, \otimes)$ and $(Y, \boxtimes)$ be groups. We call a function $f : X \to Y$ a **group homomorphism** if*

$$f(x_1 \otimes x_2) = f(x_1) \boxtimes f(x_2) \tag{33}$$

*for all $x_1, x_2 \in X$. Equivalently, f is a **group homomorphism** if the following diagram commutes*

$$\begin{array}{ccc} X \times X & \xrightarrow{f \times f} & Y \times Y \\ \downarrow{\scriptstyle \otimes} & & \downarrow{\scriptstyle \boxtimes} \\ X & \xrightarrow{f} & Y \end{array}$$

where $f \times f$ is to be read as f acting on the two elements of the product $X \times X$

Groups are objects of central importance in physics and chemistry, as we use them to encode the notion of symmetry. If you have a mathematical object which possesses some symmetry (in a sense in which I will soon define) is due to the manifestation in some form of an abstract group in the structure of the object.

4 Topology

Now that we understand sets and how to manipulate them, we want to start putting some more interesting extra structure on them. A standard reference text for point-set topology, the focus of the first part of the course, is *Topology* by James Munkres. Here are some miscelaneous links

- Experimental Topology (Video Link)
- Topology of the hole in a hole in a hole
- Topological Bagel Surgery

4.1 Axioms of a Topological Space

Definition 29. *Let X be a set. Let $\mathcal{T} = \{U_i \mid U_i \subset X\}$ be some collection of subsets of X such that*

- *$\mathcal{T}$ is closed under arbitrary unions*
- *$\mathcal{T}$ is closed under finite intersections*
- *$X, \emptyset \in \mathcal{T}$*

*Then we call $\mathcal{T}$ a **topology** on X. A set with a topology on it, $(X, \mathcal{T})$, is called a **topological space**.*

Definition 30. *Let $(X, \mathcal{T})$ be a topological space. Given some subset $U \subseteq X$, if $U \in \mathcal{T}$ then we call U an **open set** in the topology $\mathcal{T}$.*

You should take care not to confuse the open sets used to define topologies with the open intervals on $\mathbb{R}$ we discussed earlier.

There is another way of defining $\mathcal{T}$ that helps make the axioms more transparent. Instead of selecting a collection of subsets which satisfy the axioms, we can instead start with any collection (including $\emptyset$ and the set itself) and build out of that a topology which satisfies the axioms

Definition 31. *Basis Let X be a set and $\mathcal{B} = \{B_i \mid B_i \subset X\}_{i \in I}$ a collection of subsets. We can construct a topology on X by forming the set of all possible finite intersections and arbitrary unions out of $\mathcal{B}$. This collection then satisfies the axioms of a topological space by construction. The initial set $\mathcal{B}$ is thus called a* ***basis*** *for $\mathcal{T}$.*

When the exercise sheets refer to the *standard topology on* $\mathbb{R}$, that refers to the topology on $\mathbb{R}$ generated by all the open intervals on $\mathbb{R}$. For an example of what the generating of a more *discrete* topology looks like, see Figure 4. Below we describe some common topologies.

It is possible to put different topologies on the same space. Different topologies can have different levels of granularity and so give you different topological information about the space.

4.2 Topological Structures

We saw in set theory where were a number of binary operations we could perform on sets, once endowed with a topology, there are several useful *unuary* operations that are useful, especially when it comes to discussing *closed sets*

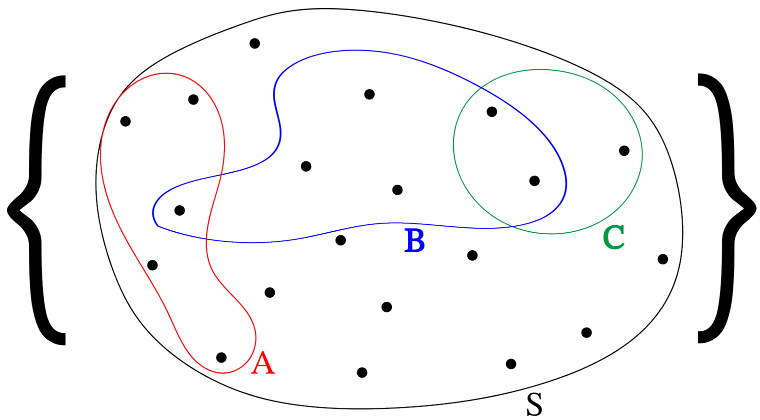
Definition 32. *Let $(X, \mathcal{T})$ be a topological space and $S \subset X$ some subset of X. We call S* ***closed*** *if $S^c \in \mathcal{T}$.*

Since we know that for any $S \subset X$ what $(S^c)^c) = S$ this definition we can also see that a set is open if and only if its complement is closed. This leads to the following concept.

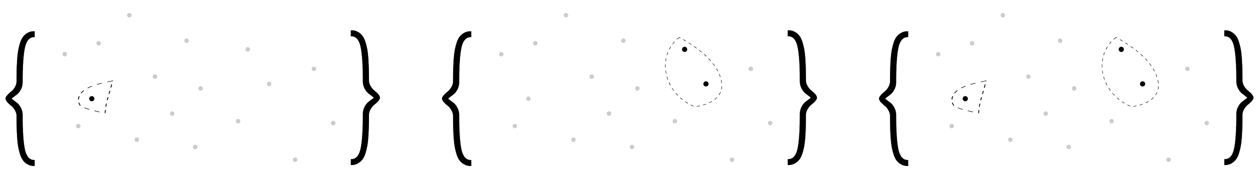
Definition 33. *Let $(X, \mathcal{T})$ be a topological space and $S \subset X$ some subset of X which is both closed and open. We call S a* ***clopen*** *set.*

TBC: Closures and interiors

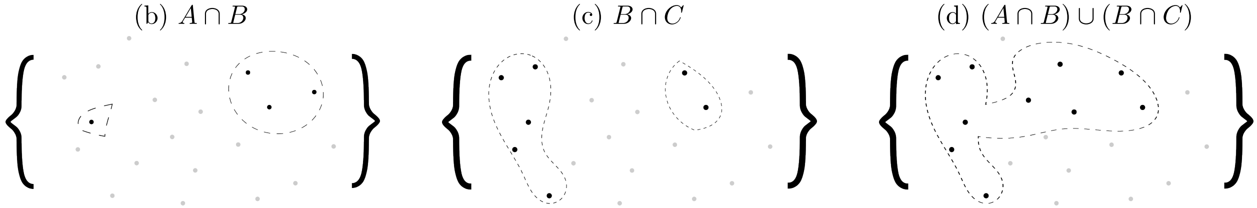
4.3 Continuous Maps



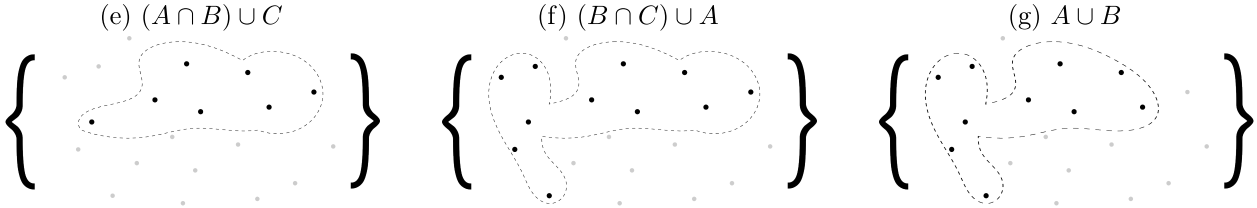
(a) A set S along with a collection of subsets $A, B, C \subset S$ which we will use to generate a topology on S



(b) $A \cap B$ (c) $B \cap C$ (d) $(A \cap B) \cup (B \cap C)$



(e) $(A \cap B) \cup C$ (f) $(B \cap C) \cup A$ (g) $A \cup B$



(h) $B \cup C$ (i) $A \cup B \cup C$ (j) $A \cup B$

Figure 4: (a) specifies the generating set for the topology on S, $(b-j)$ are the subsets that are generated using unions and intersections. The open sets are thus S itself, the empty set $\emptyset$, the subsets A, B, C and the subsets $(b-j)$