An Introduction to the Geometry of Quantisation
Accompanying Notes

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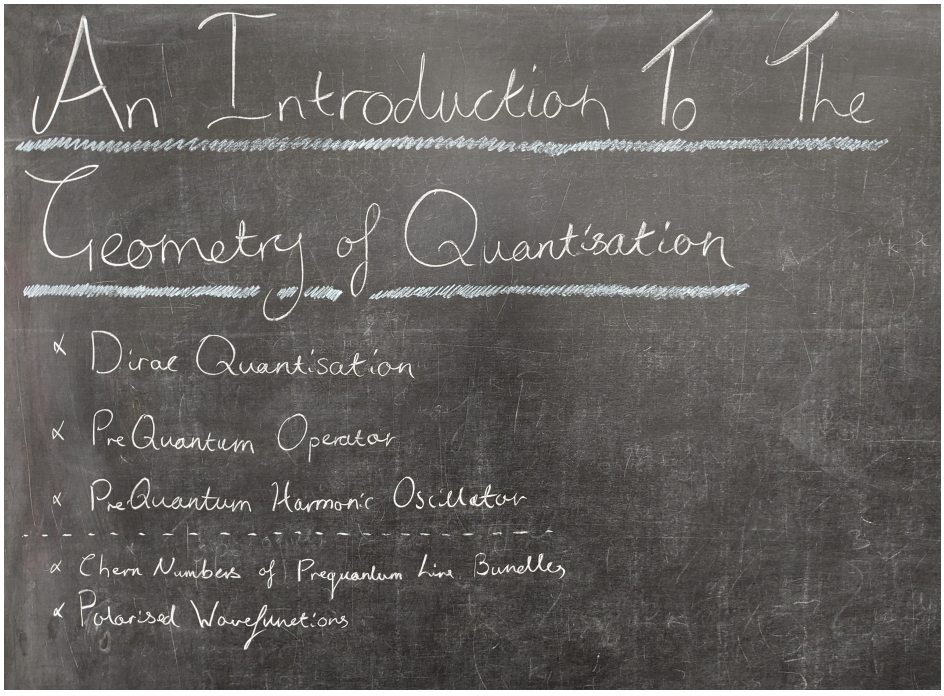


1 What is Quantisation: The Dirac Axioms

Given a *particular model* for classical mechanics and a *particular model* of quantum mechanics (including fields), how can we build a quantum mechanical system on top of a classical system?

Most well-studied case is the canonical quantization, in 1D

$$q \longmapsto \hat{q}$$
$$p \longmapsto -i\hbar\frac{\partial}{\partial q}$$

Importantly, these operators obey

$$[\hat{q}, \hat{p}] = i\hbar.$$

This is quantisation of the simplest dynamical quantities, for which Dirac wrote down a generalisation of in 1925 (see [Dir25] and [Dir81] for the general case).

Definition 1 (Dirac Conditions)**.** *Given a classical mechanics supporting the notion of an observable*[1] *f of phase space M (i.e. a smooth $f : M \to \mathbb{R}$, where $F = C^\infty(M)$ are smooth functions) and a quantum mechanics given by operators $\mathcal{O}$ on a (Hilbert) space of wavefunctions $\mathcal{H}$, a quantisation Q is a map*

$$Q : C^\infty(M) := F \to \mathcal{O}(\mathcal{H}) : f \longmapsto Q(f) = \hat{f},$$

where canonical transformations of the classical theory are mapped onto unitary transformations of the quantum theory. Such a map Q is said to be Dirac *if it obeys the following conditions:*

1. *$f \mapsto \hat{f}$ is linear*
2. *Q maps constants to constants, i.e. $r = \hat{r}\ \forall r \in \mathbb{R}$*
3. *Poisson brackets are mapped to commutators: $[\hat{f}, \hat{g}] = i\hbar Q(\{f, g\})\ \forall f, g \in C^\infty(M)$*

Recall that

$$\{A, B\} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}$$

and so a partially evaluated Poisson bracket behaves like a vector field.

Things to note about the Dirac quantisation are that it is very *symmetry focused*, it appears *local*[2]. There are other kinds of quantization, usually with conditions on operator ordering, e.g. Weyl, Wick and Anti-Wick.

Our goal is to show the problems and processes involved in **extending** *canonical quantisation* to curved and topologically nontrivial spaces. This is the theory of *Geometric Quantisation*

The quantisation scheme we will discuss **does not** have any operator ordering specification

There is a famous theorem which strongly limits the Dirac quantization, see [Got99] for the full argument.

Theorem 1. *There is no Dirac quantisation of arbitrary polynomials in x and p.*

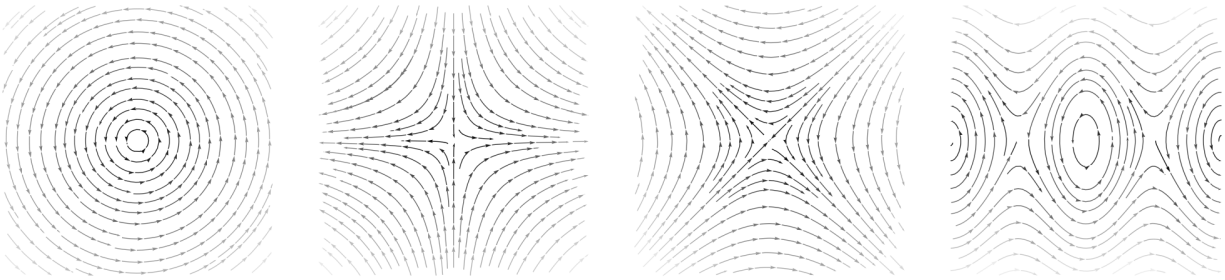
Proof. Lifted from [Hal13] theorem 13.3. The polynomial q^2p^2 has multiple representations

$$q^2p^2 = \frac{1}{9}\{q^3,p^3\} = \frac{1}{3}\{q^2p, qp^2\}$$
$$\implies i\hbar Q\left(q^2p^2\right) = \frac{1}{9}[Q\left(q^3\right), Q\left(p^3\right)] = \frac{1}{3}[Q\left(q^2p\right), Q\left(qp^2\right)]$$
$$\implies i\hbar Q\left(q^2p^2\right) = -\frac{2}{3}(-i\hbar)^3 = -\frac{1}{3}(-i\hbar)^3$$

□

[1] A smooth dynamical quantity we can evaluate along a trajectory

[2] quantum mechanics will be determined at most by the dynamics in the neighbourhood of the corresponding point in classical phase space



(a) (b) (c) (d)

Figure 1: Examples of Hamiltonian vector fields X_H for (a) the SHO, (b) the Berry-Keating Hamiltonian, (c) the inverted SHO and (d) the simple pendulum.

One can avoid running into the *GVH theorem* by restringing the set of classical observables to be quantised to certain well-behaved Lie subalgebras of observables.

We want to undertake the challenge of quantisation, building everything up from the classical phase space, only introducing additional structure when it is necessary to do so.

2 The Euclidean Kostant-Souriau Prequantum Operator

Let us take for the moment our working space of wavefunctions to be square-integrable functions over phase space

We now need a way to lift observables from classical phase space to operators on these wavefunctions. One way to do this is using the so-called *pre-quantum operator*. See the canonical reference by Woodhouse [Woo92] for a fuller discussion.

Definition 2. *Let $f, H \in C^\infty(M)$, the **Hamiltonian vector field** associated with f is written X_f and is given by (depending on sign conventions) by a variety of expressions*

$$X_H(f) = \{H, f\} = \omega(X_f, X_H) = \frac{df}{dt}$$

Vector fields have commutation relations given by the Lie derivative, for X, Y vector fields and a smooth function f this is

$$[X, Y](f) = X(Y(f)) - Y(X(f)) = L_X Y(f),$$

so why not just lift the vector fields straight away?

In particular we can show that

$$[X_f, X_g] = X_{\{f,g\}}$$

and so the objects

$$\hat{f} = i\hbar X_f$$

which commutes like

$$[\hat{f}, \hat{g}] = [i\hbar X_f, i\hbar X_g] = (i\hbar)\,(i\hbar X_{f,g})\,.$$

This appears to work well but does not satisfy our requirement of constants being mapped to multiplication operators, which is important for the behaviour of eigenvalues in the reduced phase space picture.

A simple extension of our previous map that fixes this could be

$$\hat{f} = i\hbar X_f + f$$

but this destroys the commutation property since

$$\begin{aligned}[\hat{f}, \hat{g}] = [i\hbar X_f + f, i\hbar X_g + g] &= (i\hbar)^2 \left(X_{\{f,g\}} + [X_f, g] + [f, X_g] + [f, g] \right) \\ &= i\hbar \left(i\hbar X_{f,g} \right)\end{aligned}$$

and

$$Q(\{f, g\}) = i\hbar X_{\{f,g\}} + \{f, g\}$$

so

$$[\hat{f}, \hat{g}] \neq i\hbar Q\left(\{f, g\} \right),$$

we are missing a factor of $i\hbar\{f, g\}$.

We can introduce a third and final term into our definition of Q designed to get rid of this Poisson bracket. Recalling that in terms of the symplectic form $\omega = dp \wedge dq$ we can write the Poisson bracket as

$$\{f, g\} = \omega(X_g, X_f),$$

we pick a *potential* for ω, i.e. a 1-form θ such that $d\theta = \omega$, for example $\theta = pdq$ or $\theta = \frac{1}{2}(pdq - qdp)$. Then we can define

$$\hat{f} = i\hbar \underbrace{\left(X_f - \frac{i}{\hbar}\theta(X_f) \right)}_{\text{Covariant Derivative}} + \underbrace{f}_{\text{Constant}}$$

where we claim these two terms together constitute a covariant derivative, which we give the notation

$$\hat{f} = i\hbar \nabla_{X_f} + f.$$

The commutator property can then be shown

$$\begin{aligned}[\hat{f}, \hat{g}] &= [i\hbar \nabla_{X_f} + f, i\hbar \nabla_{X_g} + g] \\ &= (i\hbar)^2 \nabla_{X_{\{f,g\}}} + i\hbar\{f, g\}\end{aligned}$$

which is consistent with

$$i\hbar Q(\{f, g\}) = i\hbar \left(i\hbar \nabla_{X_{\{f,g\}}} + \{f, g\} \right)$$

and so the covariant derivative-like object

$$Q(f) = i\hbar \nabla_{X_f} + f$$

seems to give a Dirac quantisation, but we know that there will be problems with the order of functions we can plug into it, or equivalently, the irreducibly of the resulting operators.

There is a $U(1)$ gauge freedom in defining ∇ that allows us some degree of control over the form of the wavefunctions.

2.1 Prequantum Harmonic Oscillator

We have

$$H = \frac{1}{2}(p^2 + q^2)$$

and compute

$$X_H = q\partial_p - p\partial_q$$

and so

$$\hat{H} = i\hbar(q\partial_p - p\partial_q)$$

which has eigenvalues

$$\psi_n = f\left(\sqrt{q^2+p^2}\right) e^{-in \arctan p/q},$$

best viewed in cylindrical coordinates where f is a square integral function and $n \in \mathbb{Z}$ is the eigenvalue.

We get the spectrum spacing roughly correct (no zero point energy) but the eigenfunctions are too loosely defined. In particular our eigenfunctions are functions of q *and* p, when we would expect them to be only functions of q or p separately.

2.2 Curvature in the space of wavefunctions

A more interesting topological note comes from the identifying of a covariant derivative in the construction of the prequantum operator. Computing

$$curv(\nabla)(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = -\frac{i}{\hbar}\omega(X,Y)$$

we find that our connection has **curvature!** So there is a possibility for our quantisation procedure to be **topologically nontrivial!**.

The fact that our wavefunctions depend on all variables on phase space is itself a very intriguing problem. In standard formalism it is solved by introducing a technical object called a *polarization*, which is the subject of much modern research.

2.3 Towards a full quantisation

- Describe the process of polarising
- Pairing maps between polarisations
-

3 The Topology of Phase Space: *Where do the wavefunctions live?*

We are most used to doing quantum mechanics over flat phase spaces. Generally speaking we should be prepared to do quantum mechanics on the tangent space of whatever arbitrary configuration space we confine out particle to, for example:

$$Q = \text{Euclidean Space} \xrightarrow{Phase\ Space} T^*\mathbb{R}^n \qquad Q = \text{Sphere Space} \xrightarrow{Phase\ Space} T^*S^2$$

Geometry is important when doing quantum mechanics even for flat phase spaces, as imposing constraints (selecting an eigenvalue) is often equivalent to localising phase space flows to some curved submanifold of the full phase space, often with a nontrvial topology. We will now explore how these considerations affect our ability to build wavefunctions

On spaces with nontrivial topology one typically looses the ability to speak of "functions" global objects. This is a consequence of the fact that we need more than 1 coordinate system to describe such spaces and so objects called *transition functions* introduce an extra degree of freedom in constructing our space of functions. The tools used to do this are called vector bundles and we will describe them below.

Definition 3. *A d-dimensional smooth manifold M is a space where around each point $p \in M$ I can find a neighbourhood of p, i.e. some $p \in U \subseteq M$, such that $U \cong \mathbb{R}^d$ - we say M is locally Euclidean. Such a neighbourhood $U \subseteq M$ and the function $\Phi : U \to \mathbb{R}^n$ is called a coordinate chart. The function Φ is called a local trivialisation*[3]

All the coordinate system on flat space (Cartesian, Cylindrical, Elliptic, ect.) can then be pulled back to patches on the manifold. For example, any point in $\mathbb{R}^n$ is in the neighbourhood of a global Cartesian coordinate system.

To put coordinates on the sphere one needs to split it into two hemispheres, $H^{\uparrow}$ and $H^{\downarrow}$ with local trivialisations

$$\Phi^{\uparrow} : H^{\uparrow} \to \mathbb{R}^2$$

and

$$\Phi^{\downarrow} : H^{\downarrow} \to \mathbb{R}^2$$

respectively, then make sure that wherever they overlap the change-of-variables (transition functions) are smooth, in the case of the sphere we want

$$\Phi^{\downarrow} \circ (\Phi^{\uparrow})^{-1} : \mathbb{R}^2|_{overlap} \to H^{\uparrow} \cap H^{\downarrow} \cong S^1 \to \mathbb{R}^2|_{overlap}$$

and the corresponding inverse to be smooth bijections.

Definition 4. *A rank k vector bundle over a manifold M is a collection (E, M, π) of three pieces of information. E is called the total space, M is called the base space and $\pi : E \to M$ is called the fiber-wise projection. The bundle places the structure of the vector space $\mathbb{R}^k$ above each point in the base space (called the **fiber**), i.e. $\pi^{-1}(p) \cong \mathbb{R}^k$ for all $p \in M$. Every local trivialisation of $U \subset M$ induces a local trivialisation of the bundle, for example $\Phi_U : U \times \mathbb{R}^k \to \pi^{-1}(U)$*

If a manifold is a space which at any point looks locally like $U \cong \mathbb{R}^n$, then a rank k vector bundle E is a space which at every point looks locally like $U \times \mathbb{R}^k$.

If we have a rank 1 vector bundle over a manifold M, then the fiber $\mathbb{R} \cong \pi^{-1}(p)$ above each point should be though of as the space in which scalar functions evaluated at p takes its value. Our regular notion of a function is thus only well-defined on an individual local trivialisation of a bundle of appropriate dimension. We thus consider functions to be the local trivialisation of a global coordinate-independent object called a *section* of the bundle.

[3] For those unfamiliar with the general pattern of differential geometry, often we are mapping small areas of manifolds to flat space and then trying to pullback all the normal Euclidean calculus and geometry back through Φ

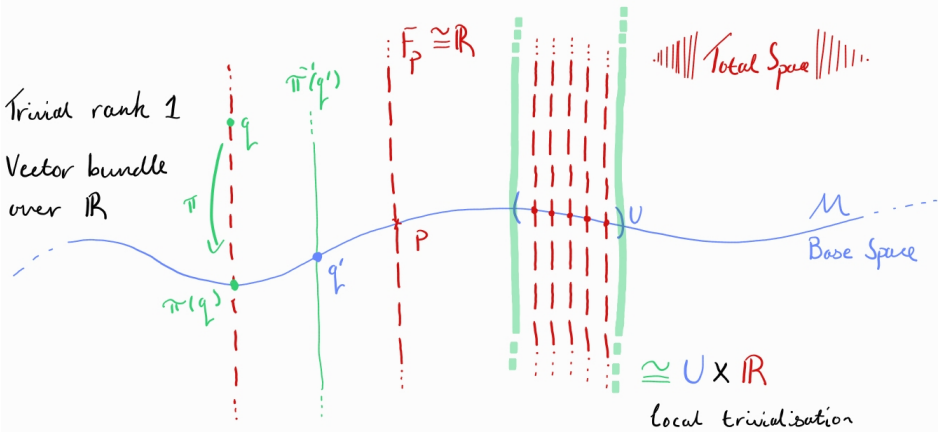


Figure 2: Caption

Definition 5. *Let (E, M, π) be a rank k vector bundle over M. A section of E, usually written $s \in \Gamma(E)$, is a map $s : M \to E$ such that $\pi(s(p)) = p$ for all $p \in M$, i.e. a section maps only into the fiber above a the point in question.*

Just as local trivialisation of a manifold are related by transition functions, the local trivialisation of bundles are also related by maps, consider the case of the clutching construction on S^2 shown in Figure 3.

Our complex scalar fields describing our quantum wavefunctions are thus going to be sections of a rank 1 *complex* vector bundles, also known as a complex line bundle. Complex line bundles are completely classified by the *Chern number*. We see that when quantising curved spaces we need to *choose* the Chern number of the bundle in which our wavefunctions live. Complex lines over flat phase space only have Chern number 0.

You *could* argue for making the choice to favour the trivial complex line bundle as the canonical choice, but theory of *Geometric Quantisation* tells us that nontrivial bundles are actually better to work with[4]. In fact, conditions on the curvature of the bundle containing the wavefunctions will force us to consider nontrivial bundles all the time.

4 Cutting Down Phase Space: *Polarised Wavefunctions*

Pre to Full Quantization

References

[Dir25] Paul Adrien Maurice Dirac. "The fundamental equations of quantum mechanics". In: *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* 109.752 (1925), pp. 642–653.

[Dir81] Paul Adrien Maurice Dirac. *The principles of quantum mechanics*. 27. Oxford university press, 1981.

[4] In particular, *Holomorphic* and *Kähler* bundles are commonly used.

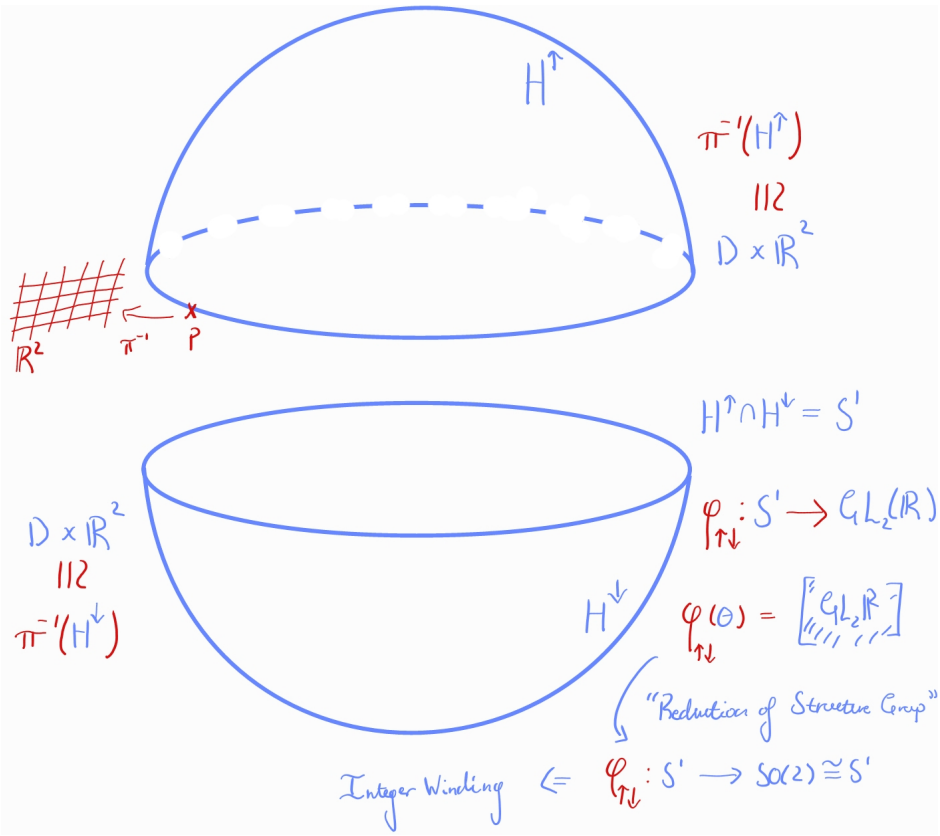


Figure 3: The clutching construction for a rank 2 real vector bundle (or rank 1 complex vector bundle) over S^2

[Woo92] Nicholas Michael John Woodhouse. *Geometric quantization*. Oxford university press, 1992.

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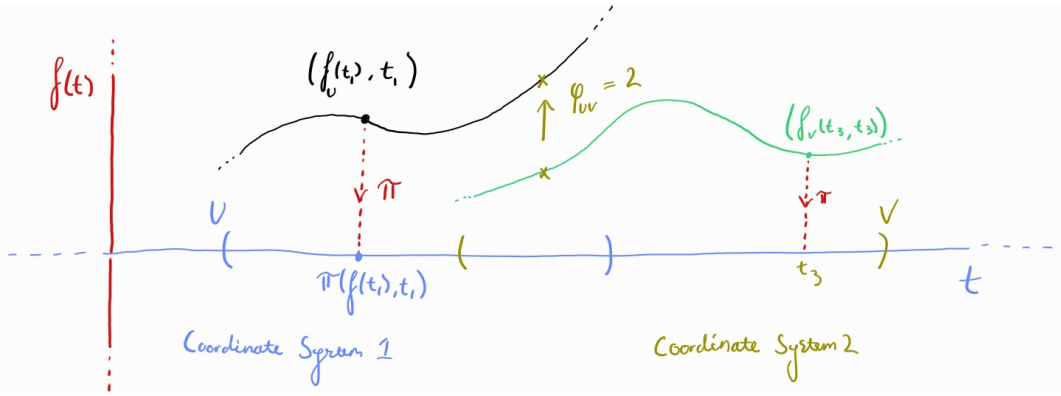
[Hal13] Brian C Hall. *Quantum theory for mathematicians*. Springer, 2013.



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